Stochastic resonance in a spatially extended system

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We have analyzed the phenomenon of stochastic resonance in a spatially extended system by studying a simple version of a bistable reaction-diffusion model. Knowledge of the nonequilibrium potential for this system allows us to determine, using the albedo boundary condition as our (modulated) control parameter, the bistability region in the nonequilibrium potential, the probability for the decay of the metastable extended states, and approximate expressions for the correlation function and the signal-to-noise ratio in this case. [S1063-651X(96)51709-5]

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The phenomenon of *stochastic resonance* is characterized by the enhancement of the signal-to-noise ratio (SNR) caused by injection of noise into a periodically modulated nonlinear system. The increase in the noise intensity from initial small values induces an increase in the SNR ratio until it reaches a maximum, beyond which there is a decay of SNR for large noise values [1–4]. Some recent reviews and conference proceedings clearly shows the wide interest of this phenomenon and the state of the art [5].

What is still under study are the features of this phenomenon in spatially extended systems. Some early studies were related to the nucleation of solitons and kinks [6], globally coupled nonlinear oscillators [7], recent numerical simulation of arrays of coupled nonlinear oscillators [8] and excitable systems [9], and even some preliminary results regarding homogeneous solutions within the context of the time dependent Ginzburg-Landau equation [10].

In this paper we present an analysis of the stochastic resonance phenomenon in a spatially extended system, exploiting previous results obtained using the notion of the *nonequilibrium potential* [11] in the context of a simple reaction-diffusion model. The specific model we shall focus on, with a known form of the Lyapunov function, corresponds to a one-dimensional, one-component model [12,13] mimicking general bistable reaction-diffusion models [12]. Besides the study of the role of boundary conditions (BC's) in pattern selection [14,15], we were particularly concerned with the *global stability* of the resulting nonhomogeneous structures [16]. Such analysis was carried out by exploiting the notion of *nonequilibrium potential* or *Lyapunov functional* of the system [11].

The particular adimensional form of the model that we work with is [14–18]

$$\partial_t \phi = \partial_{yy}^2 \phi - \phi + \phi_h \theta(\phi - \phi_c). \tag{1}$$

We have considered here a class of stationary structures $\phi(y)$ in the bounded domain $y \in (-y_L, y_L)$ with albedo boundary conditions at both ends, $d\phi/dy|_{y=\pm y_L} = \mp k\phi(\pm y_L)$, where $k>0$ is the albedo parameter. These are the spatially symmetric solutions to Eq. (1) already studied in Ref. [14]. The explicit forms of these stationary patterns are given in Eqs. (9) and (3) of Refs. [14] and [17], respectively.

The double-valued coordinate y_c, at which $\phi = \phi_c$, is given by

$$y_c^{\pm} = \frac{1}{2} y_L - \frac{1}{2} \ln\left[\frac{z\gamma(k,y_L) \pm \sqrt{z^2\gamma(k,y_L)^2 + 1 - k^2}}{1+k}\right], \tag{2}$$

with $\gamma(k,y) = \sinh(y) + k\cosh(y)$, and $z = 1 - 2\phi_c/\phi_h$ $(-1 < z < 1)$.

When $y_c^{\pm}$ exists and $y_c^{\pm} < y_L$, this solutions represents a structure with a central ''hot'' zone $(\phi > \phi_c)$ and two lateral ''cold'' regions $(\phi < \phi_c)$. For each parameter set there are two stationary solutions, given by the two values of y_c. In Ref. [14], it has been shown that the structure with the smallest ''hot'' region [with $y_c = y_c^+$ denoted by $\phi_u(y)$] is unstable, whereas the other one [with $y_c = y_c^-$ denoted by $\phi_1(y)$] is linearly stable. The trivial homogeneous solution $\phi = 0$ (denoted by ϕ_0) exists for any parameter set and is always linearly stable. These two linearly stable solutions are the only stable stationary structures under the given albedo boundary conditions. We will concentrate on the region of values of the parameters z, y_L, and k where both nonhomogeneous structures exist.

For our system with the albedo BC that we are considering here, the Lyapunov functional (LF) reads [16]

$$\mathcal{F}[\phi,k] = \int_{-y_L}^{y_L} \left\{ -\int_0^{\phi(y,t)} [-\phi' + \phi_h \theta(\phi' - \phi_c)] d\phi' + \frac{1}{2}[\partial_y \phi(y,t)]^2 \right\} dy + \frac{k}{2} \phi(y,t)^2 |_{\pm y_L}. \tag{3}$$

Replacing the explicit forms of the stationary nonhomogeneous solutions [see, for instance, Eq. (9) in Ref. [14]], we obtain the explicit expression [17]

$$\mathcal{F}^{\pm} = -\phi_h^2 y_c^{\pm} z + \phi_h^2 \sinh(y_c^{\pm}) \frac{\gamma(k, y_L - y_c^{\pm})}{\gamma(k, y_L)}, \tag{4}$$

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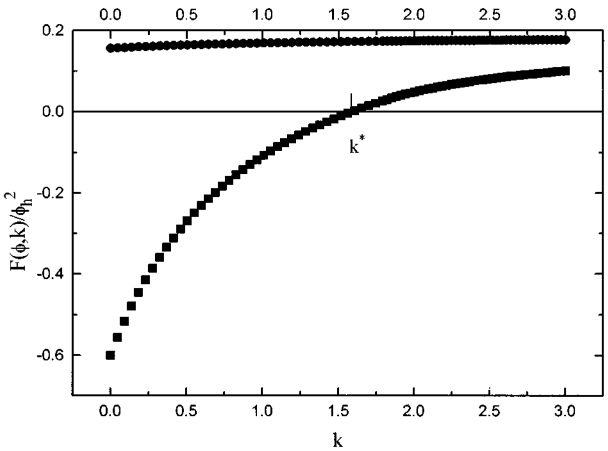

FIG. 1. Values of the nonequilibrium potential $\mathcal{F}$ as a function of the albedo parameter k, for fixed value of the length $(y_L=2)$ and the ratio ϕ_c/ϕ_h $(\phi_c/\phi_h=0.35)$. The bottom curve corresponds to the potential of the stable pattern $\phi_1(y)$, while the top one indicates the potential of the unstable pattern $\phi_u(y)$. The potential for the stable homogeneous pattern $\phi_0(y)$, coincides with the horizontal axis. We have indicated the point $k=k^*=1.5889\ldots$, where both stable structures $[\phi_0$ and $\phi_1(y)]$ have the same stability.

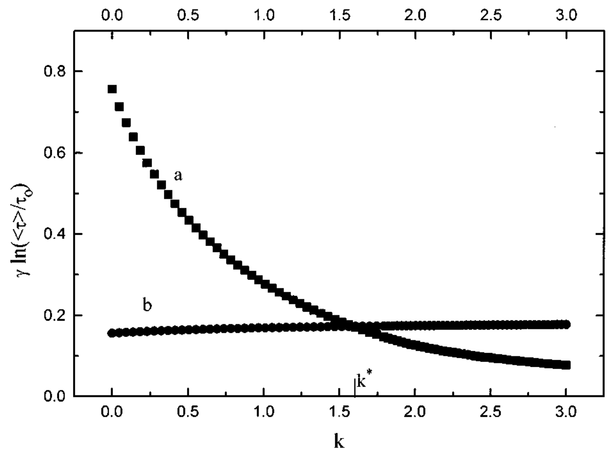

FIG. 2. Values of $\gamma\ln(\langle\tau\rangle/\tau_0)$ as a function of the albedo parameter k, for the same parameter values as in Fig. 1. Line (a) indicates the decay (transition) from the pattern $\phi_1(y)$ towards ϕ_0, while line (b) corresponds to the inverse decay (transition). The point where $k=k^*$, is also indicated.

while for the homogeneous trivial solution $\phi_0=0$, we have instead $\mathcal{F}[\phi_0,k]=0$.

In Fig. 1 we have plotted the Lyapunov functional $\mathcal{F}[\phi]$ as a function of k for a fixed system size, $y_L=2$, and a fixed value of the ratio ϕ_c/ϕ_h (i.e., fixed value of z). The curves correspond to the nonhomogeneous structures, $\mathcal{F}^{\pm}$, whereas the horizontal line stands for the Lynpunov functional (LF) of the trivial solution. We have focused on the bistable zone, the upper branch being the LF of the unstable structure, where $\mathcal{F}$ attains a maximum, while in the lower branch (for $\phi=\phi_0$ or $\phi=\phi_1$), the LF has local minima.

It is important to note that, since the LF for the unstable solution ϕ_u is always positive and, for the stable nonhomogeneous structure ϕ_1, $\mathcal{F}<0$ for $k\to 0$, the LF for this structure vanishes for an intermediate value $k=k^*$ of the albedo parameter. At that point, the stable nonhomogeneous structure $\phi_1(y)$ and the trivial solution $\phi_0(y)$ exchange their relative stability. We will work in the bistable region in the neighborhood of the point $k=k^*$.

In order to account for the effect of fluctuations, we need to include in the time-evolution equation of our model [Eq. (1)] a fluctuation term, modeled as an additive noise source [12], yielding a stochastic partial differential equation for the random field $\phi(y,t)$:

$$\partial_t\phi(y,t)=\partial_{yy}^2\phi-\phi+\phi_h\theta(\phi-\phi_c)+\xi(y,t). \qquad (5)$$

We make the simplest assumptions about the fluctuation term $\xi(y,t)$, i.e., that it is a Gaussian white noise with zero mean value and a correlation function given by $\langle\xi(y,t)\xi(y',t')\rangle=2\gamma\delta(t-t')\delta(y-y')$, where γ denotes the noise strength.

At this point we will use a recently developed scheme that describes the decay of extended metastable states [19]. Following such a scheme, and in order to obtain the transition probability between metastable and stable states, it is necessary to find the conditional probability for the random field $\phi(y,t)$ to have the value $\phi_{stable}(y,t)$ at time t, given that at the initial time $t=0$ the system was in a state $\phi_{meta}(y,0)$. This probability can be represented by a path integral over those realizations of the random field $\xi(y,t)$ that satisfy the initial and final conditions, that is,

$$P[\phi_{stable}(y,t)|\phi_{meta}(y,0)]\sim\int\mathcal{P}[\xi]\delta(\phi(y,t)-\phi_{meta}(y,0))\mathcal{D}\xi(y,t), \qquad (6)$$

where the statistical weight $\mathcal{P}[\xi]$ for a Gaussian white noise is of the form

$$\mathcal{P}[\xi]\sim\exp\left[-\frac{1}{2\gamma}\int_0^t dt\int_{-y_L}^{y_L}dy\,\xi^2(y,t)\right]. \qquad (7)$$

In the limit of small noise intensity, the main contribution in Eq.(6) is given by the realizations of the random field close to the most probable trajectory [11,19]. This fact allows us to estimate the result of Eq. (6) by the steepest-descent method. This procedure yields the following Kramers's like result for the first-passage-time $\langle\tau\rangle$ [20]:

$$\langle\tau\rangle=\tau_0\exp\left\{\frac{\Delta\mathcal{F}[\phi,k]}{\gamma}\right\}, \qquad (8)$$

where $\Delta\mathcal{F}[\phi,k]=\mathcal{F}[\phi_{unst}(y),k]-\mathcal{F}[\phi_{meta}(y),k]$. The prefactor τ_0 is usually determined by the curvature of $\mathcal{F}[\phi,k]$ at its extrema (minima) and is typically several orders of magnitude smaller than the average time $\langle\tau\rangle$. The behavior of $\langle\tau\rangle$ as a function of the albedo parameter k is shown in Fig. 2, for the same values of the system parameters as in Fig. 1.

We assume now that, due to an external harmonic variation of the reflectivity at the boundaries, the parameter k has an oscillatory part: $k(t)=k^*+\delta k\cos(\Omega t+\varphi)$. For the space extended problem, we need to evaluate the space-time correlation function $\langle \phi(y,t)\phi(y',t')\rangle$, which is given by the double functional integral

$$\langle \phi(y,t)\phi'(y',t')\rangle = \int \mathcal{D}[\phi]\int \mathcal{D}[\phi']\phi(y,t) \times \phi'(y',t')P[\phi(y,t)|\phi'(y',t'),\varphi] \times P_{as}(\phi(y,t),\varphi), \tag{9}$$

where $P_{as}(\phi(y,t),\varphi)$ indicates the asymptotic form of the probability distribution of $\phi(y,t)$ (depending ''parametrically'' on φ), while $P[\phi(y,t)|\phi'(y',t'),\varphi]$ is the conditional probability [analogous to Eq. (6)]. For $t'=t+T$, t large and $T\to\infty$, we can assume the asymptotic result $P[\phi(y,t)|\phi'(y',t'),\varphi]\sim P_{as}(\phi'(y',t'),\varphi)$ [3]. Hence

$$\langle \phi(y,t)\phi'(y',t')\rangle_{T\to\infty} \sim \langle \phi(y,t),\varphi\rangle_{as}\langle \phi'(y',t'),\varphi\rangle_{as}, \tag{10}$$

with

$$\langle \phi(y,t),\varphi\rangle_{as} = \int \mathcal{D}[\phi]\phi(y,t)P_{as}(\phi(y,t),\varphi). \tag{11}$$

In the present case, it is necessary to make a double Fourier transform of the correlation function in order to obtain, instead of the power spectrum, the *generalized susceptibility* $S(\kappa,\omega)$ [12] as

$$S(\kappa,\omega) = \int dy \int dt\, e^{i[\kappa y-\omega t]}\langle \phi(y,t),\varphi\rangle_{as}^2. \tag{12}$$

To evaluate it we will use a simplified point of view, based on the ideas of Ref. [2]. In particular, due to the bistable character of our problem potential, we can almost straightforwardly apply the results of their Sec. V. To proceed with the calculation of the correlation function we need to evaluate the transition probabilities

$$W_{\pm} = \tau_0^{-1}\exp\left\{ -\frac{\Delta\mathcal{F}[\phi,k]}{\gamma}\right\} \tag{13}$$

where

$$\Delta\mathcal{F}[\phi,k] = \Delta\mathcal{F}[\phi,k^*] + \delta k\left(\frac{\partial\Delta\mathcal{F}[\phi,k]}{\partial k}\right)_{k=k^*}\cos(\Omega t+\varphi). \tag{14}$$

This yields for the transition probabilities

$$W_{\pm} \simeq \frac{1}{2}\left(\alpha_0 \mp \alpha_1\frac{\delta k}{\gamma}\cos(\Omega t+\varphi)\right), \tag{15}$$

where $\alpha_0 \approx \exp(-\Delta\mathcal{F}[\phi,k^*])$ and $\alpha_1 \approx \alpha_0\,(d\Delta\mathcal{F}/dk|_{k*})$. With this identification, and using the fact that $\phi_0^{as}(y)=\phi_0=0$, in the equation analogous to Eq. (3.10) (in Ref. [2]), only one term is left. Hence, after averaging over the random phase φ, we end with an expression similar to Eq. (3.12) (also in Ref. [2]) but where we shall replace $c^2=\phi_1(y)^2$.

To complete the calculation we need to perform the Fourier transform of the correlation function in time as well as in space in order to obtain the generalized susceptibility $S(\kappa,\omega)$. Due to the decoupling of the correlation function shown in Eq. (10), the generalized susceptibility also decouples adopting the form

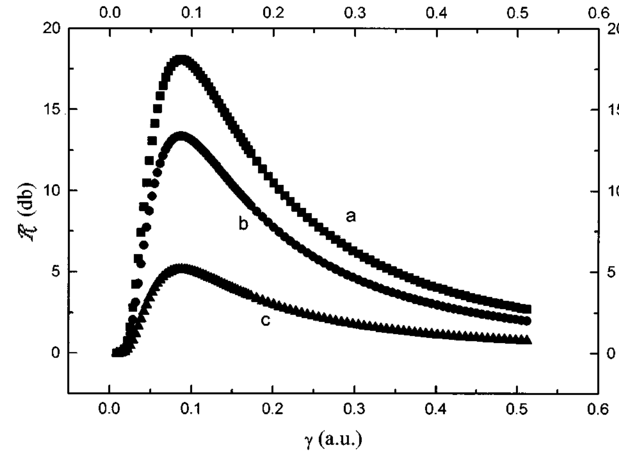

FIG. 3. Curves of $\mathcal{R}(\kappa)$ as a function of the noise parameter γ [see Eq. (17)], for different values of $\kappa(=n\pi/2y_L)$. We have normalized with the component of $F(k)$ corresponding to $\kappa=0$. We adopted $k=k^*$ and the same parameter values as in Fig. 1. (a) $\kappa=0$ $(n=0)$, (b) $\kappa=0.785$ $(n=1)$, (c) $\kappa=1.57$ $(n=2)$.

$$S(\kappa,\omega) = F(\kappa)S(\omega), \tag{16}$$

where $F(\kappa)\sim\int dy\, \exp(i\kappa y)\phi_1(y)^2$, and $S(\omega)$ is the usual power spectrum function, as given in Eq. (5.7) of Ref. [2]. The result for the SNR, $\mathcal{R}$, that now becomes a function of κ is

$$\mathcal{R}(\kappa) \sim F(\kappa)(\Lambda\lambda\gamma^{-1})^2\exp(-2\Delta\mathcal{F}[\phi,k^*]/\gamma), \tag{17}$$

where λ is the previously indicated linear eigenvalue, and $\Lambda\sim d\Delta\mathcal{F}/dk|_{k*}\delta k$.

This configures the main result of the present work. Equation (17) is analogous to what has been obtained in zero-dimensional systems, but now including a prefactor with a dependence on the wavelength κ. In Fig. 3 we show the dependence of the present SNR on κ, for typical values of the parameters (same as in Figs. 1 and 2). In fact, as we are considering a bounded system, we have a discrete Fourier spectra.

From Eq. (1), it is clear that we have scaled the length with the diffusion constant. This diffusion constant is a measure of the coupling between different spatial units, hence, a variation in our system length will imply an inverse variation of the coupling constant. In the present case, a numerical analysis indicates that such a variation does not show a significant variation in the SNR of the system. The present result does not agree with other recent results, based on numerical simulations, indicating an enhancement of the response of a stochastic resonator by its coupling into a chain of identical resonators [8]. However, this is not a surprise as the situation described here differs from the one discussed in Ref. [8]. Here we have that the coupling to the external modulated field is present, effectively, only at the boundaries, while the situation described in Ref. [8] corresponds to the modulation of every one of the coupled nonlinear oscillators. In this regard, a preliminary analysis of a situation similar to the one described in Ref. [8] within the present model [that is, keeping k constant and introducing the modulation directly into Eqs. (1) or (5), for instance, by modulation of the threshold parameter ϕ_c] shows evidence of an enhancement of the SNR due to coupling [21]. However this is not conclusive as we need to make a more careful analysis of the problem, in order to reach a more complete picture of the space dependent phenomena. As indicated by the present rough calculation (see Fig. 3), we expect different strengths in the stochastic resonance phenomena for different wavelengths, as the dependence of the generalized susceptibility $S(\kappa,\omega)$ on κ and ω—that will not necessarily factorize—will also imply that $\mathcal{R}\sim\mathcal{R}(\kappa,\omega)$.

An important conclusion to be drawn from the identification of the Lyapunov functional with a ''thermodynamical-like potential'' is that near the point k^* at which two states have the same stability, the problem seems to admit a one-dimensional analog [17,18]. This feature is in contrast with the infinite-dimensional character of the whole function space, and has been used to strongly simplify the analysis of our system.

We expect that the present form of analysis could be extended to activator-inhibitor or multicomponent systems such as those studied in Ref. [22]. The possible applications in chemical [23] and biological systems [5,24,25], and its relation with spatiotemporal synchronization problems [26,8], are very well known. It is worth remembering that activator-inhibitor systems have a tight connection with Bonhoffer–van der Pol–like nonlinear spatially coupled oscillators. All these points will be the subject of further work.

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[1] R. F. Fox, Phys. Rev. A**39**, 4148 (1989).
[2] B. McNamara and K. Wiesenfeld, Phys. Rev. A**39**, 4854 (1989).
[3] P. Jung and P. Hänggi, Phys. Rev. A**44**, 8032 (1991).
[4] M. Dykman, R. Manella, P. V. E. McClintock, N. D. Stein, and N. Stocks, Phys. Rev. A**47**, 1629 (1993); M. Dykman, D. G. Luchinsky, R. Manella, P V. E. McClintock, N. D. Stein, and N. Stocks, J. Stat. Phys.**70**, 463 (1993).
[5] F. Moss, in *Some Problems in Statistical Physics*, edited by G. Weiss (SIAM, Philadelphia, 1992); P. Jung, Phys. Rep. **234**, 175 (1993); *Proceedings of NATO Advanced Research Workshop on Stochastic Resonance in Physics and Biology*, edited by F. Moss, A. Bulsara, and M. Shlesinger, in J. Stat. Phys. **70**, No. 1/2 (1993); K. Wiesenfeld and F. Moss, Nature (London) **373**, 33 (1995); *Proceedings of the 2nd Internationl Workshop on Stochastic Resonance, Elba, Italy, 1994*, edited by R. Mannella and P. V. E. McClintock, in Il Nuovo Cimeato D **17**, (1995).
[6] P. Hänggi, F. Marchesoni, and P. Sodano, Phys. Rev. Lett. **60**, 2563 (1988); F. Marchesoni, Phys. Rev. Lett. **73**, 2394 (1994).
[7] A. Bulsara and G. Schmera, Phys. Rev. E **47**, 3734 (1993); P. Jung, U. Behn, E. Pantazelou, and F. Moss, Phys. Rev. A **46**, R1709 (1992).
[8] J. F. Lindner, B. K. Meadows, W. L. Ditto, M. E. Inchiosa, and A. Bulsara,Phys. Rev. Lett. **75**, 3 (1995).
[9] P. Jung and G. Mayer-Kress, Phys. Rev. Lett. **74**, 208 (1995).
[10] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **18**, 2239 (1985).
[11] R. Graham, in *Weak Noise Limit and Nonequilibrium Potentials of Dissipative Dynamical Systems*, in *Instabilities and Nonequilibrium Structures*, edited by E. Tirapegui and D. Villaroel (D. Reidel, Dordrecht, 1987); R. Graham and T. Tel, Phys. Rev. A **42**, 4661 (1990).
[12] H. S. Wio, *An Introduction to Stochastic Processes and Non-equilibrium Statistical Physics* (World Scientific, Singapore, 1994), Chap. 5; A. S. Mikhailov, *Foundations of Synergetics I* (Springer-Verlag, Berlin, 1990).
[13] W. J. Skocpol, M. R. Beasley, and M. Tinkham, J. Appl. Phys. **45**, 4054 (1974); B. Rossand J. D. Lister, Phys. Rev. A **15**, 1246 (1977); D. Bedeaux, P. Mazur, and R. A. Pasmanter, Physica A **86**, 355 (1977); D. Bedeaux and P. Mazur, *ibid.* **105**, 1 (1981).
[14] C. L. Schat and H. S. Wio, Physica A **180**, 295 (1992).
[15] S. A. Hassan, M. N. Kuperman, H. S. Wio, and D. H. Zanette, Physica A **206**, 380 (1994); S. A. Hassan, D. H. Zanette, and H. S. Wio, J. Phys. A **27**, 5129 (1994); S. A. Hassan and D. H. Zanette, Physica A **214**, 435 (1995).
[16] G. Izús, R. Deza, O. Ramírez, H. Wio, D. Zanette, and C. Borzi, Phys. Rev. E **52**, 129 (1995).
[17] D. H. Zanette, H. S. Wio, and R. Deza, Phys. Rev. E **53**, 353 (1996).
[18] F. Castelpoggi, H. S. Wio, and D. H. Zanette (unpublished).
[19] A. Foster and A. S. Mikhailov, Phys. Lett. A **126**, 459 (1988); S. P. Fedotov, *ibid.* **176**, 220 (1993).
[20] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[21] H. S. Wio and F. Castelpoggi, in Stochastic Resonance in a Reaction-Diffusion System: Enhancement Due to Coupling, Proc. Conf. UPoN '96, Szeged, 1996, edited by C. R. Doering, L. B. Kiss, and M. Shlesinger (World Scientific, Singapore, in press).
[22] G. Drazer and H. S. Wio (unpublished).
[23] D. S. Leonard and L. E. Reichl, Phys. Rev. E **49**, 1734 (1994).
[24] S. M. Bezrukov and I. Vodyanoy, Nature (London) **378**, 362 (1995).
[25] J. J. Collins, C. C. Chow, and T. T. Imhoff, Nature (London) **376**, 236 (1995).
[26] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (Springer-Verlag, Berlin, 1984).